Quasi-Stationary Distributions for Markov Chains

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Outline of Topics

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QSDs

- A Quasi-Stationary Distribution (in short QSD) for X is a probability measure supported on $(0, \infty)$ satisfying for all $t \geq 0$,

$$\mathbf{P}_{\nu}(\mathbf{X}(\mathbf{t}) \in \mathbf{A} | \mathbf{T} > \mathbf{t}) = \nu(\mathbf{A}), \ \forall \ \mathbf{borel} \ \ \mathbf{set} \ \ \mathbf{A} \subseteq (\mathbf{0}, \infty).$$

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  for some $\mu \in (0, \infty)$.
- A QSD must be infinitely divisible (D.Vere-Jones 1969)

QSDs

- A probability measure π supported on $(0, \infty)$ is a LCD If there exists a probability measure ν on $(0, \infty)$ such that the following limit exists in distribution

$$\lim_{t \to \infty} \mathbf{P}_\nu(\mathbf{X}(\mathbf{t}) \in \bullet \mid \mathbf{T} > \mathbf{t}) = \pi(\bullet).$$

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  We also say that ν is attracted to π or is in the domain of attraction of π or π is a ν-LCD
- Trivially, any QSD ν is an ν-LCD.
- The ν-LCD is a QSD (Vere-Jones(1969)).

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- (i) determination of all QSD's; and
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- (iii) The rate of convergence of the transition probabilities of the conditioned process to their limiting values.

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- For a Birth and death process, Erik A. VAN DOORN (Adv.Appl.Prob.23, 683-700,1991) obtained
  Proposition (i) if $S = \infty$, then either λ_C (Kingman's decay parameter)=0 and there is no QSD, or $\lambda_C > 0$ and there is a one-parameter family of QSDs, Viz,$\{\{q_j(x)\}, 0 < x \leq \lambda_C\}$.
  (ii) If $S < \infty$, then $\lambda_C > 0$ and there is precisely one QSD,Viz,$\{q_j(\lambda_C)\}$

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QSDs of Markov Chains

- Let X_t be a continuous-time Markov chain in $I = \{0\} \cup \{1, 2, \ldots\}$ such that 0 is an absorbing state. Let $C \equiv \{1, 2, \ldots\}$. Denote by $Q = (q_{ij})$ the q-matrix (transition rate matrix) and $P(t) = (P_{ij}(t))$ the transition function. X_t is stochastically monotone if and only if $\sum_{j \geq k} P_{ij}(t)$ is a nondecreasing function of i for every fixed $k \in I$ and $t > 0$. We assume that all states other than 0 form an irreducible class and that Q is totally stable, conservative and regular, that is, $q_i = \sum_{i \neq j} q_{ij} < \infty$, and the minimal process $\{X_t\}_{t \geq 0}$corresponding to Q is an honest process. We further define $T = \inf\{t \geq 0 : X_t = 0\}$, the absorption time at 0. So $X_t = 0$ for any $t \geq T$.

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QSDs of Markov Chains

- For continuous-time general Markov chains, P.A.Ferrari, H.Kesten,S.Martinez, and P.Picco (The Annals of Probability 1995,Vol.23, No.2,501-521.) obtained
  Proposition 2 Assume that

$$\lim_{i\to\infty} P_i(T < t) = 0 \quad \text{for any} \quad t \geq 0$$

  and that $P_i(T < \infty) = 1$ for some (and hence all) i. Then a necessary and sufficient condition for the existence of a QSD is that

$$E_i(e^{\theta T}) < \infty$$

  for some $\theta > 0$ and some $i \in C$ (and hence for all i).

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QSDs of Markov Chains

- Our result is

  Theorem 1 Assume that

$$\lim_{i\to\infty} E_i T = \infty,$$

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$$E_i(e^{\theta T}) < \infty$$

for some $\theta > 0$ and some $i \in C$ (and hence for all i). When it holds, There exists a family of QSDs.

QSDs of Markov Chains

- For stochastically monotone Markov chains, we discuss the existence, uniqueness and domain of attraction of QSDs.
  Theorem 2 Assume Q is regular and conservative, and X_t is stochastically monotone,
  (i) If $\lim_{i\to\infty} E_i T = \infty$, then there exists a QSD if and only if
  $$E_i(e^{\theta T}) < \infty$$
  for some $\theta > 0$ and some $i \in C$ (and hence for all i).
  (ii) If $\lim_{i\to\infty} E_i T < \infty$, and the set $N_0 = \{i \in C : q_{i0} > 0\}$ is finite, then there is a unique QSD .Moreover, the unique QSD $\rho = \{\rho_j, j \in C\}$ attracts all initial distributions that supported in C, that is, for any probability measure $\nu = \{\nu_i, i \in C\}$, $\rho_j = \lim_{t\to\infty} \mathbf{P}_\nu(\mathbf{X_t = j | T > t}), \quad \mathbf{j \in C}$.

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Thank you all for your attention!